

Discriminant Varieties for Stick Knots and Links

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August 7, 2026

Abstract

How many knot types can be built from a fixed budget of straight sticks? We prove the explicit factorial-scale window

$$N^{(2/3+o(1))N} \leq |\mathcal{K}_N| \leq N^{3N+o(N)}.$$

A recent strong-geometry reconstruction theorem already implies an $N^{O(N)}$ upper bound after standard sign-pattern counting; extracting only the signs used in its proof gives $N^{6N+o(N)}$. Our direct discriminant argument halves that leading upper exponent: it turns polygonal self-intersection into a sparse real-algebraic chamber problem in only linearly many dimensions. A Garside refinement of the Malyutin–Stupakov pure-braid injection gives the $2/3$ lower exponent, already among prime satellite knots. The result ties knot topology directly to real algebraic geometry, fewnomial structure, and permutation combinatorics.

1 Introduction

Let $p_1, \dots, p_N \in \mathbb{R}^3$ be labeled vertices joined cyclically by straight segments. The space of embedded N -gons is the complement of a self-intersection discriminant. Its connected components are sometimes called geometric knot types: two polygons in the same component are connected by an isotopy through N -gons. Calvo computed this space for $N = 6, 7$, showing in particular that geometric equivalence is finer than ordinary topological knot type [8]. The exact number of components is unknown in general.

The semialgebraic starting point is also already present in Calvo’s work: for every pair of non-incident edges, the possible-intersection locus is described there as a codimension-one cubic semialgebraic piece, with additional inequalities selecting intersections of the segments rather than merely their supporting lines. Thus neither the polygon-space discriminant nor its semialgebraicity is claimed as new here. Our upper bound is a quantitative extraction from that model: we retain the binomial dependence on the number of walls in a refined sign-condition component estimate.

The purpose of this note is to isolate what can be proved by elementary semialgebraic methods. The basic facts are:

- the discriminant is semialgebraic and has explicit low-degree chart equations;
- knot type is constant on each chamber of the complement;
- every knot type with stick number at most N occurs in at least one chamber of positive measure; and
- determinant-related sign-condition bounds, with their binomial factor retained, give

$$|\mathcal{K}_N| \leq (AN)^{3N} = N^{3N+o(N)};$$

- combined with the factorial arc-index lower bound of Malyutin–Stupakov, the chamber estimate makes explicit the logarithmic growth order $\log |\mathcal{K}_N| = \Theta(N \log N)$, equivalently the scale $N^{\Theta(N)}$;
- a further extraction from the Malyutin–Stupakov pure-braid construction, using Garside normal forms and an explicit factorial-size family of doubly alternating permutations, sharpens the lower side to $|\mathcal{K}_N| \geq N^{(2/3+o(1))N}$, already among prime satellite knots; and
- after summing over component decompositions, there are at most $N^{4N+o(N)}$ ordinary links with at most N total sticks.

The determinant form of the crossing walls is the reason that one expects a substantially structured chamber-count problem rather than an arbitrary arrangement of high-degree hypersurfaces.

A generic projection of an N -stick polygon has at most $N(N-3)/2$ crossings, and classical exponential enumeration by crossing number gives the baseline $|\mathcal{K}_N| \leq \exp(O(N^2))$ [13]. More recently, Gros–Ramirez Alfonsin proved that the strong geometry of a generic vertex configuration determines its polygonal-knot type [15]. Their result, combined with standard fixed-degree sign-pattern counting, already implies $|\mathcal{K}_N| = N^{O(N)}$: counting the full wedge-chirotope table gives $N^{33N+o(N)}$, while counting only the $O(N^3)$ degree-at-most-six sign queries used in their reconstruction proof gives $N^{6N+o(N)}$. These counting consequences do not appear as stated theorems in their paper, but they are immediate enough that we treat the factorial upper scale as prior implicit mathematics.

Our contribution is the much sharper direct stick-space estimate obtained by making Calvo’s semialgebraic discriminant model quantitative:

$$|\mathcal{K}_N| \leq N^{(3+o(1))N},$$

which improves the reconstruction-based leading exponent from 6 to 3. The Garside construction below simultaneously raises the explicit lower exponent to $2/3$. The present result is therefore a sharper quantitative synthesis of established geometric and sign-condition inputs, not a new discriminant encoding or a new general real-algebraic theorem. Randell–Simon–Togle directly studied the counting function by polygon length and proved an exponential lower bound, but no comparable factorial lower bound [22]. Millett later listed the corresponding question for equilateral polygons as an open problem and reported that no useful estimate by edge number was then available [20, Problem 5.2]. Recent computational work concerns upper bounds for individual stick numbers rather than this counting function [9].

2 Normalized stick-knot parameter space

Let

$$H = \mathbf{1}^\perp \subset \mathbb{R}^N.$$

After subtracting the centroid of the vertices, the three coordinate rows of a full-dimensional polygon lie in H . Applying an invertible linear change of coordinates in $\mathbb{R}^3$, one may normalize the row Gram matrix to the identity. This leads to the Stiefel model

$$X_N = \text{St}_3(H) = \{Q \in \mathbb{R}^{3 \times N} : Q\mathbf{1} = 0, QQ^T = I_3\}.$$

The columns $q_i = Qe_i \in \mathbb{R}^3$ are the normalized vertices. Since $\dim H = N - 1$,

$$\dim X_N = 3(N - 1) - \frac{3 \cdot 4}{2} = 3N - 9.$$

Quotienting by $SO(3)$ gives the oriented Grassmannian

$$\widetilde{\text{Gr}}_3(H) = X_N/SO(3), \quad \dim \widetilde{\text{Gr}}_3(H) = 3N - 12.$$

This does not change chamber counts: the quotient is a locally trivial bundle with connected fiber $SO(3)$, and rotations move a polygon inside the same embedding chamber. This normalization is geometrically natural, although the upper bound below is proved directly in the unsliced vertex space. Quotienting by all of $O(3)$, on the other hand, identifies mirror images.

3 Local equations for crossing walls

Consider two non-incident directed edges ab and cd , and write

$$v = p_b - p_a, \quad w = p_d - p_c, \quad r = p_c - p_a.$$

In the xy -projection chart, set

$$D = \det \begin{pmatrix} x_b - x_a & x_d - x_c \\ y_b - y_a & y_d - y_c \end{pmatrix},$$

$$N_\lambda = \det \begin{pmatrix} x_c - x_a & x_d - x_c \\ y_c - y_a & y_d - y_c \end{pmatrix}, \quad N_\mu = \det \begin{pmatrix} x_c - x_a & x_b - x_a \\ y_c - y_a & y_b - y_a \end{pmatrix}.$$

When $D \neq 0$, the two projected supporting lines meet at parameters

$$\lambda = \frac{N_\lambda}{D}, \quad \mu = \frac{N_\mu}{D}.$$

The projected crossing lies in the interiors of both segments precisely when

$$N_\lambda(D - N_\lambda) > 0, \quad N_\mu(D - N_\mu) > 0.$$

The height difference at the crossing is

$$h = z_a + \lambda(z_b - z_a) - z_c - \mu(z_d - z_c).$$

Clearing denominators gives

$$F_{ab,cd} = D(z_a - z_c) + N_\lambda(z_b - z_a) - N_\mu(z_d - z_c).$$

Equivalently, up to sign,

$$F_{ab,cd} = \det(p_b - p_a, p_d - p_c, p_c - p_a).$$

This is the affine orientation determinant of the four vertices p_a, p_b, p_c, p_d , up to sign. Thus each over/under wall is not merely a low-degree equation: it is one of the determinant walls governing the order type of the vertex configuration. As a polynomial in the coordinate variables, it has at most 24 monomial terms before cancellation. The same construction is used in the xz - and yz -projection charts. The cases where the relevant 2×2 projected determinant vanishes are lower-dimensional chart degeneracies. They may be included in a full chartwise sign description, but are not needed for the enlarged coordinate-free wall family used in Theorem 2.

4 The discriminant and its chambers

Let $\Delta_N \subset X_N$ be the set for which the cyclic piecewise-linear map is not an embedding. Thus an edge may collapse, adjacent edges may meet away from their common endpoint, or two combinatorially non-incident edges may meet. The last condition says that, for some such pair ab, cd , there are $\lambda, \mu \in [0, 1]$ with

$$p_a + \lambda(p_b - p_a) = p_c + \mu(p_d - p_c).$$

Edge collapse and adjacent-edge overlap have similarly elementary polynomial descriptions. These first-order formulas use only polynomial equalities and inequalities with existentially quantified segment parameters, so the Tarski–Seidenberg theorem [2, Section 2.2] implies that Δ_N is semi-algebraic. This formulation includes the adjacent backtracking degeneracy that is missed if one mentions only non-incident edge pairs.

Define

$$C_N = b_0(X_N \setminus \Delta_N),$$

the number of embedding chambers. For a knot type K , let $C_N(K)$ be the number of chambers in which the polygon has type K .

Proposition 1 (Chambers carry one knot type). *The ambient isotopy class of the cyclic polygon is constant on each connected component of $X_N \setminus \Delta_N$.*

Proof. Connected semialgebraic sets are semialgebraically path-connected, so two points of a chamber can be joined by a path in $X_N \setminus \Delta_N$. Such a path is a one-parameter family of embedded polygonal curves with fixed cyclic combinatorics. For a sufficiently fine subdivision of the parameter interval, consecutive polygons are close enough that the straight-line interpolation of corresponding vertices remains embedded. This gives an isotopy through embedded polygonal curves, and the ambient isotopy extension theorem [10] gives ambient isotopy of the endpoints. $\square$

Remark 1 (Generic walls). At a smooth codimension-one point of a single wall $F_{ab,cd} = 0$, with $D \neq 0$, $0 < \lambda, \mu < 1$, and no simultaneous degeneracy, crossing the wall changes the sign of the height difference $h = F_{ab,cd}/D$. Thus the two adjacent diagrams differ by one over/under switch. This is a local statement; it does not imply that all crossing assignments in a large diagram are independently realizable.

5 Support of the knot type map

Let $\text{stick}(K)$ denote the stick number of a knot type K . This is the minimum number of straight segments needed in a polygonal representative of K ; see, for example, the standard tables and references in KnotInfo [18] and the stick-number work of Adams et al. [1].

Theorem 1 (Support theorem). *For $N \geq 4$, the image of the knot type map*

$$\tau_N : X_N \setminus \Delta_N \longrightarrow \{\text{ambient isotopy classes of knots}\}$$

is exactly

$$\mathcal{K}_N = \{K : \text{stick}(K) \leq N\}.$$

Moreover, if $K \in \mathcal{K}_N$, then $\tau_N^{-1}(K)$ contains a nonempty open subset of X_N . In particular it has positive measure for the natural $O(N-1)$ -invariant Riemannian volume on X_N , and

$$C_N(K) \geq 1 \iff \text{stick}(K) \leq N.$$

Proof. If $Q \in X_N \setminus \Delta_N$, then its columns form an embedded polygon with N sticks. Hence the represented knot type $K = \tau_N(Q)$ satisfies $\text{stick}(K) \leq N$.

Conversely, suppose $\text{stick}(K) \leq N$. Choose a polygonal representative of K with at most N sticks and subdivide edges, if necessary, to obtain exactly N labeled vertices. After an arbitrarily small ambient isotopy, assume that the centered coordinate rows span a three-dimensional subspace of H . Let $M \in \mathbb{R}^{3 \times N}$ be the matrix of centered vertex coordinates. Then $M\mathbf{1} = 0$ and MM^T is positive definite. Put

$$A = (MM^T)^{1/2}, \quad Q = A^{-1}M.$$

Then $Q\mathbf{1} = 0$ and $QQ^T = I_3$, so $Q \in X_N$. The original centered polygon is the image of the Q -polygon under the orientation-preserving linear map A . Therefore Q represents the same knot type K , and $Q \notin \Delta_N$.

Finally, stability of PL embeddings under sufficiently small vertex perturbations [10] gives a neighborhood of Q in X_N representing the same knot type. Concretely, non-incident edges have positive separation, every edge is nonzero, and local injectivity at the finitely many adjacent pairs persists; at a straight subdivision vertex the two outgoing directions have strictly negative dot product. Nonempty open subsets of the compact smooth manifold X_N have positive natural $O(N-1)$ -invariant Riemannian volume. $\square$

6 A determinantal semialgebraic upper bound

Theorem 2 (Stick-knot upper bound). *There exists an absolute constant $A > 0$ such that, for every integer $N \geq 4$,*

$$|\mathcal{K}_N| \leq (AN)^{3N}.$$

Consequently $|\mathcal{K}_N| \leq N^{3N+o(N)}$ as $N \rightarrow \infty$.

Proof. Write $e_i = p_{i+1} - p_i$, with indices read cyclically. There are

$$r_N = \frac{N(N-3)}{2}$$

unordered pairs of non-incident edges. For each such pair put

$$f_{ij} = \det(e_i, e_j, p_j - p_i),$$

and, for each adjacent pair, put

$$h_i = \|e_i \times e_{i+1}\|^2.$$

The f_{ij} have degree three and the h_i have degree four. If none vanishes, consecutive edges are nonzero and noncollinear, hence meet only at their prescribed common endpoint. Two non-incident segments cannot meet either: an intersection would make their two directions and displacement coplanar and force the corresponding f_{ij} to vanish. Thus the complement of these

$$s = r_N + N = \frac{N(N-1)}{2}$$

algebraic walls consists entirely of embedded polygons. It is harmless that these enlarged walls also delete benign coplanar configurations and straight subdivisions.

Let $G_N \subset (\mathbb{R}^3)^N \cong \mathbb{R}^{3N}$ be the complement of this enlarged wall family. Every knot in $\mathcal{K}_N$ has a representative in G_N : subdivide to exactly N vertices and make an arbitrarily small generic perturbation. PL stability preserves knot type, while a generic perturbation avoids the finitely

many nonzero wall polynomials. On the other hand, every point of G_N is an embedded polygon by the preceding paragraph.

The knot-type map on G_N is constant on path components. Indeed, a path in G_N is a one-parameter family of embedded polygonal curves, and ambient isotopy extension identifies its endpoints. Since G_N is open in a Euclidean space, connected components and path components agree. The induced map

$$\pi_0(G_N) \longrightarrow \mathcal{K}_N$$

is therefore surjective, and

$$|\mathcal{K}_N| \leq b_0(G_N).$$

On G_N the sign of every wall is locally constant, so $b_0(G_N)$ is the sum of the component counts of its realizable strict sign conditions. Dimension-sensitive cell bounds retaining the essential $(s/m)^m$ dependence go back to Pollack–Roy [23]. We use the part of the sign-condition theorem that is lost in the cruder s^m notation. The explicit Barone–Basu bound [4, Theorem 1.1], specialized to s polynomials of degree at most four on $\mathbb{R}^m$, implies, after absorbing fixed-degree factors into an absolute constant B ,

$$b_0(G_N) \leq B^m \sum_{j=0}^m 4^j \binom{s+1}{j}.$$

Explicitly, take $\mathcal{Q} = \emptyset$, $V = \mathbb{R}^m$, $k' = k = m$, and $d = 4$ in their formula (with any fixed vacuous d_0 , say 2). Its j -th auxiliary factor is bounded using $\binom{m+1}{j+1} \leq 2^{m+1}$, with all remaining fixed-degree factors absorbed into B^m .

For exact agreement with the Lean development, index a redundant family of

$$S = N + N^2$$

slots: use the N adjacent walls above, put the appropriate determinant in each relevant ordered-pair slot, and put the constant polynomial 1 in every irrelevant slot. This leaves the nonzero locus unchanged. With $m = 3N$, one has $m \leq (S+1)/2$ for $N \geq 5$, and hence

$$\sum_{j=0}^m 4^j \binom{S+1}{j} \leq (m+1)4^m \binom{S+1}{m} \leq (m+1) \left(\frac{4e(S+1)}{m} \right)^m.$$

Since $(S+1)/m = O(N)$, enlarging the absolute constant absorbs the factor $m+1$ and gives $b_0(G_N) \leq (AN)^{3N}$. Enlarging A also covers $N = 4$. This proves the first estimate. $\square$

Corollary 1 (Comparison with geometric chambers). *For the normalized polygon space of the preceding sections,*

$$|\mathcal{K}_N| \leq \sum_{K \in \mathcal{K}_N} C_N(K) = C_N.$$

Proof. By Theorem 1, every $K \in \mathcal{K}_N$ is represented by at least one chamber. By Proposition 1, each chamber represents only one topological knot type. Therefore the chamber count bounds the number of represented knot types from above. $\square$

Theorem 3 (Asymptotic growth order). *As $N \rightarrow \infty$,*

$$\log |\mathcal{K}_N| = \Theta(N \log N),$$

or equivalently $|\mathcal{K}_N| = N^{\Theta(N)}$.

Proof. The upper bound follows from Theorem 2. For the lower bound, Malyutin–Stupakov prove that the number of oriented prime-knot classes with arc index at most k is at least

$$\left\lfloor \frac{k+4}{11} \right\rfloor !$$

for $k > 5$ [19, Theorem 1]. Take $k = \lfloor 2N/3 \rfloor + 1$. The Huh–Oh inequality [16, Theorem 3] gives

$$\text{stick}(K) \leq \frac{3}{2}(k-1) = \frac{3}{2} \left\lfloor \frac{2N}{3} \right\rfloor \leq N.$$

Thus every resulting knot is within the stick budget, and forgetting orientation costs at most a factor of two. Hence

$$|\mathcal{K}_N| \geq \frac{1}{2} \left\lfloor \frac{\lfloor 2N/3 \rfloor + 5}{11} \right\rfloor ! = N^{(2/33+o(1))N}.$$

Stirling’s formula and the upper estimate $|\mathcal{K}_N| \leq N^{3N+o(N)}$ complete the proof. $\square$

Remark 2 (Upper bounds for links). Let $\mathcal{L}_N$ be the set of ambient isotopy classes of ordinary unoriented unordered links in $\mathbb{R}^3$ that admit a polygonal representative with at most N total sticks. The same semialgebraic upper-bound mechanism gives, after summing over component decompositions, an estimate

$$|\mathcal{L}_N| \leq N^{4N+o(N)}.$$

Indeed, a fixed labeled link combinatorics is a cyclic decomposition of the N labeled vertices, equivalently a permutation of those vertices up to the irrelevant choices of starting point on each cycle. Thus there are at most $N!$ labeled component decompositions to consider. For each such decomposition, the number of pairs of non-incident edges is at most $N^2/2$, and there are exactly N cyclically adjacent edge pairs across all components. Add the corresponding N squared-cross-product walls to the non-incident determinant walls. The same argument as in Theorem 2 then gives at most

$$(AN)^{3N}$$

semialgebraic chambers for that fixed decomposition. Every unmarked unordered link type with total stick number at most N is represented in at least one of these labeled models after subdividing edges and making an arbitrarily small full-dimensional perturbation. Passing from labeled component data to ordinary unordered links can only identify types, not create new ones. Therefore

$$|\mathcal{L}_N| \leq N! (AN)^{3N} = N^{4N+o(N)}.$$

The link case is not literally the single cyclic-polygon Stiefel model: one must also account for the number of components and their cyclic decompositions.

Remark 3 (Upper bounds versus lower bounds). The chamber-to-knot-type map is surjective onto $\mathcal{K}_N$, but it is not injective. Many chambers can carry the same isotopy class; this already occurs in Calvo’s low- N computations [8]. Consequently an upper bound for C_N is automatically an upper bound for $|\mathcal{K}_N|$, but a lower bound for C_N is not a lower bound for the number of knot types unless one separately distinguishes the isotopy classes carried by those chambers.

7 A quantitative lower-bound refinement

Theorem 3 already settles the logarithmic growth order using the printed Malyutin–Stupakov theorem. We now use their stronger Proposition 1, which retains the number of permutation-braid factors as a parameter, to sharpen the lower bound. Combining that flexibility with a large language of Garside normal forms gives the estimate proved below. We include all combinatorial details because our argument extracts a consequence of their proposition rather than restating their printed factorial theorem.

For $w = w_1 \cdots w_n \in S_n$, write

$$\text{Des}(w) = \{i \in \{1, \dots, n-1\} : w_i > w_{i+1}\}, \quad D_n = \{1, 3, 5, \dots\} \cap \{1, \dots, n-1\}.$$

Set

$$G_n = \{w \in S_n : \text{Des}(w) = \text{Des}(w^{-1}) = D_n\}, \quad g_n = |G_n|.$$

Thus G_n is the set of doubly alternating permutations in the down–up convention.

Lemma 1 (Explicit doubly alternating family). *For every $k \geq 1$,*

$$g_{4k} \geq (k!)^4.$$

More precisely,

$$(4k)! \leq 2^{9k} (k!)^4, \quad \log g_{4k} = \log((4k)!) - O(k).$$

Proof. Use zero-based indices. Every $x \in \{0, \dots, 4k-1\}$ has unique coordinates

$$x = 2kh + 2r + p, \quad h, p \in \{0, 1\}, \quad 0 \leq r < k.$$

Choose four arbitrary permutations $\sigma_{h,p} \in S_k$, and define

$$w_\sigma(2kh + 2r + p) = 2k(1-p) + 2\sigma_{h,p}(r) + (1-h).$$

This is a permutation: in the coordinates (h, r, p) , it is the bijection

$$(h, r, p) \mapsto (1-p, \sigma_{h,p}(r), 1-h).$$

Even zero-based positions map to the upper half of the values and odd positions to the lower half. Therefore $w_1 > w_2 < w_3 > \cdots$ in one-based notation. The same statement holds for the inverse: the parity of the displayed value is $1-h$, so even values have preimages in the upper half of the positions and odd values have preimages in the lower half. Thus both w_σ and w_σ^{-1} have descent set D_{4k} .

The four permutations $\sigma_{h,p}$ can be recovered from w_σ , so the construction is injective and gives $g_{4k} \geq (k!)^4$. Finally,

$$\begin{aligned} (4k)! &= \binom{4k}{k} \binom{3k}{k} \binom{2k}{k} (k!)^4 \\ &\leq 2^{4k} 2^{3k} 2^{2k} (k!)^4 = 2^{9k} (k!)^4. \end{aligned}$$

Together with the trivial bound $g_{4k} \leq (4k)!$, this gives the claimed logarithmic sandwich. The entire construction, its injectivity, both descent conditions, the exact factorial inequality, and the $O(k)$ logarithmic deficit are verified in Lean. $\square$

Lemma 2 (Many short pure braids). *For integers $n \geq 3$ and $q \geq 1$, the pure braid group PB_n contains at least*

$$\left\lceil \frac{g_n^q}{n!} \right\rceil$$

distinct elements, each expressible as a product of at most $q + 1$ positive permutation braids.

Proof. Let A_w be the positive permutation braid inducing $w \in S_n$. Elrifai–Morton identify these braids with the simple elements in the classical Garside structure. In their convention, the starting and finishing sets are

$$S(A_w) = \text{Des}(w), \quad F(A_w) = \text{Des}(w^{-1});$$

the second identity follows from $F(A) = S(\text{rev } A)$. Their left normal-form criterion says that a sequence of simple factors is canonical when $S(A_{i+1}) \subseteq F(A_i)$, and that the resulting expression is unique [11, Lemmas 2.3 and 2.4 and Theorem 2.9].

For every $w \in G_n$, both sets above equal D_n . Since $n \geq 3$, D_n is nonempty and proper, so A_w is neither the identity nor the Garside half twist. It follows that every word

$$A_{w_1} A_{w_2} \cdots A_{w_q}, \quad (w_1, \dots, w_q) \in G_n^q,$$

is already its unique left normal form. The g_n^q tuples therefore give distinct positive braids.

Let $\rho : B_n \rightarrow S_n$ be the endpoint-permutation homomorphism. By the pigeonhole principle, at least $\lceil g_n^q/n! \rceil$ of the preceding braids have a common endpoint permutation u . Append the same positive permutation braid $A_{u^{-1}}$ to each of them. Every resulting braid is pure, and right cancellation in B_n shows that they remain distinct. If the appended braid is the identity it may be omitted; hence each result is a product of at most $q + 1$ positive permutation braids. $\square$

Theorem 4 (Finite stick-number lower bound). *For $k \geq 1$ and $q \geq 1$, put $n = 4k$ and*

$$a_{k,q} = 4k \max\{11, q + 9\} - 4, \quad s_{k,q} = \left\lfloor \frac{3}{2}(a_{k,q} - 1) \right\rfloor.$$

The number of ordinary unoriented prime satellite-knot types with stick number at most $s_{k,q}$ is at least

$$\left\lceil \frac{1}{2} \left\lceil \frac{g_{4k}^q}{(4k)!} \right\rceil \right\rceil \geq \frac{((k!)^4)^q}{2(4k)!}.$$

Proof. Malyutin–Stupakov construct, for every $n > 2$, an injection from PB_n to oriented knot types. Every knot in its image is prime and satellite, and if the input braid is a product of s permutation braids, then the image has arc index at most

$$n \max\{11, s + 8\} - 4$$

[19, Proposition 1, pp. 3–4]. Apply this injection to the braids from Lemma 2. Since their factor length is at most $q + 1$, their images have arc index at most $a_{k,q}$.

Forgetting knot orientation has fibers of size at most two, so at least the first displayed number of ordinary knot types remain. Prime satellite knots are nontrivial, and Huh–Oh prove for every nontrivial knot that

$$\text{stick}(K) \leq \frac{3}{2}(\alpha(K) - 1)$$

[16, Theorem 3]; thus all these knots have stick number at most $s_{k,q}$. The last inequality follows from Lemma 1. $\square$

We first extract the headline lower exponent in a form that will also be certified directly in Lean. Fix any real $c < 2/3$, and choose a fixed integer $q \geq 2$ so large that

$$\beta_q := \frac{2(q-1)}{3(q+9)} > c.$$

For each N , set

$$k_N = \left\lfloor \frac{N}{6(q+9)} \right\rfloor, \quad n_N = 4k_N.$$

Then the stick budget in Theorem 4 is at most N . The factorial estimate from Lemma 1, followed by the elementary lower Stirling bound, gives

$$\begin{aligned} \log |\mathcal{K}_N| &\geq (q-1) \log((4k_N)!) - 9qk_N \log 2 - \log 2 \\ &\geq 4(q-1)k_N \log(4k_N) - O_q(k_N). \end{aligned}$$

Because q is fixed,

$$k_N = \frac{N}{6(q+9)} + O_q(1), \quad \log(4k_N) = \log N + O_q(1),$$

and hence

$$\log |\mathcal{K}_N| \geq \beta_q N \log N - O_q(N) \geq cN \log N$$

for all sufficiently large N . Since this holds for every $c < 2/3$,

$$|\mathcal{K}_N| \geq N^{(2/3+o(1))N}.$$

All knots counted here are prime satellites.

For a sharper explicit remainder, let the word length vary with N . For sufficiently large N , set

$$q_N = \lfloor \log N \rfloor, \quad k_N = \left\lfloor \frac{2N+15}{12(q_N+9)} \right\rfloor, \quad n_N = 4k_N.$$

Then $q_N \geq 2$, $k_N \geq 1$, and

$$\frac{3}{2}(n_N(q_N+9) - 5) \leq N.$$

Theorem 4 and Lemma 1 therefore give prime satellite knots in $\mathcal{K}_N$ satisfying

$$\begin{aligned} \log |\mathcal{K}_N| &\geq 4q_N \log(k_N!) - \log((4k_N)!) - \log 2 \\ &\geq (q_N-1) \log((4k_N)!) - 9q_N k_N \log 2 - \log 2 \\ &= 4(q_N-1)k_N \log(4k_N) - O(q_N k_N). \end{aligned}$$

Here

$$q_N k_N = O(N), \quad 4(q_N-1)k_N = \frac{2}{3}N + O\left(\frac{N}{\log N}\right), \quad \log(4k_N) = \log N - \log \log N + O(1).$$

Consequently

$$\log |\mathcal{K}_N| \geq \frac{2}{3}N \log N - \frac{2}{3}N \log \log N - O(N),$$

which recovers the same lower exponent with an explicit second-order deficit.

Remark 4 (At most N versus an N -edge representative). These two formulations count the same knot types. Starting from any embedded polygon with fewer than N edges, subdivide an edge and replace the new collinear vertex by a sufficiently small local kink. PL stability preserves the ambient isotopy class and makes the two new incident edges genuine noncollinear sticks. Repeating gives an embedded representative with exactly N polygon edges. Hence the lower and upper bounds in this section apply both to stick number at most N and to realizability by an N -edge polygon. This is not a statement about the shell $\{K : \text{stick}(K) = N\}$; no comparable lower bound for every exact-minimal-stick shell is asserted here.

Corollary 2 (Refined asymptotic bounds). *As $N \rightarrow \infty$,*

$$N^{(2/3+o(1))N} \leq |\mathcal{K}_N| \leq N^{3N+o(N)}.$$

In particular

$$\log |\mathcal{K}_N| = \Theta(N \log N),$$

or equivalently $|\mathcal{K}_N| = N^{\Theta(N)}$. These asymptotic bounds determine the order of growth of the logarithm. They do not furnish an asymptotic equivalent for $|\mathcal{K}_N|$. Nor do they determine whether the normalized logarithm

$$\frac{\log |\mathcal{K}_N|}{N \log N}$$

has a limit.

For ordinary links the lower-bound problem is more direct. In this paragraph all links are unoriented and unordered, and

$$\mathcal{L}_N = \{L : L \text{ admits a polygonal representative with at most } N \text{ total sticks}\}.$$

The invariant used below is the matrix of absolute pairwise linking numbers, viewed up to simultaneous permutation of the components. This is an invariant of unoriented unordered ambient isotopy.

Proposition 2 (Addressed linking matrices). *There is an absolute constant $B_0 > 0$ with the following property. For every $m \geq 2$ and every symmetric matrix*

$$A = (a_{ij})_{1 \leq i, j \leq m}, \quad a_{ii} = 0, \quad a_{ij} \in \mathbb{Z}_{\geq 0},$$

there is an ordinary unoriented unordered link $L(A)$ such that

$$\text{stick}(L(A)) \leq B_0 \left(m + \sum_{i < j} a_{ij} \right),$$

and $L(A)$ and $L(A')$ are ambient isotopic only if $A = A'$.

Proof. We first prescribe an abstract absolute linking matrix and then realize it by a polygonal link. Use frame components

$$R, F_1, \dots, F_m, U, V$$

and payload components

$$P_1, \dots, P_m.$$

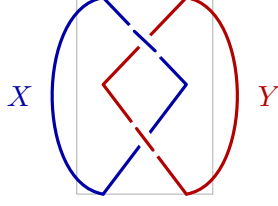


Figure 1: The componentwise closure of this tangle is a Hopf link.

The odd part of the absolute linking matrix is fixed as follows:

$$|\text{lk}(R, U)| = |\text{lk}(R, V)| = |\text{lk}(R, F_1)| = 1,$$

$$|\text{lk}(F_i, F_{i+1})| = 1 \quad (1 \leq i < m), \quad |\text{lk}(F_i, P_i)| = 1 \quad (1 \leq i \leq m),$$

and all other entries involving a frame component have even value 0. Among payload components set

$$|\text{lk}(P_i, P_j)| = 2a_{ij}.$$

Reducing the absolute linking matrix modulo 2 gives a graph. Its vertices are the components; its edges are precisely the odd linking pairs above. This graph has only one nontrivial automorphism, namely the transposition of the two leaves U, V . In particular it fixes R , then fixes the path

$$F_1, F_2, \dots, F_m$$

in order, and therefore fixes each payload component P_i . Hence an unoriented unordered ambient isotopy between two such links must identify the payload submatrix entry-by-entry. Since the payload entries are $2a_{ij}$, the matrix A is recovered from the ordinary unmarked link type.

It remains to realize the prescribed matrix with the stated stick cost. Here is a concrete routing model. Give every component a private straight rail through a long rectangular box, with the rails mutually parallel and separated. Allocate disjoint thin slabs along the rail direction, one slab for each required unit of linking. In a slab assigned to components X, Y , leave every other rail straight and replace short subsegments of the X - and Y -rails by a fixed polygonal realization of the two-strand full twist σ_1^2 , or its mirror σ_1^{-2} , as in Figure 1. Its endpoints are matched to the original rail segments on the same respective sides of the slab. The slabs and small tubular neighborhoods of all rails are disjoint. Outside the long box, close each rail by a return arc in its own separated height plane; these closing arcs can be chosen as polygonal rectangles and introduce no linking.

After choosing auxiliary orientations, choose between the two mirror-image full twists so that its two crossings have positive sign. The crossing formula

$$\text{lk}(X, Y) = \frac{1}{2} \sum_{p \in X \cap Y} \text{sign}(p)$$

shows that each inserted twist raises $\text{lk}(X, Y)$ by one. Because its supporting ball meets no other component, it changes no other pairwise linking number. Each component uses a fixed number of sticks for its rail and return arc, and each clasp replaces two straight subsegments by a fixed bounded number of sticks. Thus a diagram with q clasps has at most $c_0 m + c_1 q$ sticks for universal constants c_0, c_1 . Since the final invariant uses absolute values, the auxiliary orientations are irrelevant. The fixed odd address graph uses $O(m)$ clasps, while the payload matrix uses $2 \sum_{i < j} a_{ij}$ clasps. Enlarging one universal constant B_0 gives the claimed estimate. $\square$

Theorem 5 (Unmarked unordered link production). *There is an absolute positive integer B such that, for all $n \geq 2$,*

$$|\mathcal{L}_{Bn}| \geq \binom{\binom{n}{2} + n}{n} \geq n^n \exp(-O(n)).$$

Consequently, for some constants $c, C > 0$,

$$N^{cN} \leq |\mathcal{L}_N| \leq N^{CN}$$

for all sufficiently large N .

Proof. Take $m = n$ in Proposition 2, and consider all nonnegative symmetric payload matrices with

$$\sum_{i < j} a_{ij} \leq n.$$

There are $M = \binom{n}{2}$ independent entries, so the number of such matrices is the stars-and-bars number

$$\binom{M + n}{n}.$$

Each produces a distinct unmarked unordered link type, and each has total stick number at most Bn for a universal B . Finally,

$$\binom{M + n}{n} = \prod_{k=1}^n \frac{M + k}{k} \geq \frac{M^n}{n!} = n^n \exp(-O(n)),$$

because $M = \binom{n}{2}$. The upper bound in the final display is Remark 2; the lower bound follows by replacing n by $\lfloor N/B \rfloor$. $\square$

Thus ordinary unmarked unordered links already have the $N^{\Theta(N)}$ growth scale, with a completely elementary invariant: the absolute pairwise linking matrix, together with a parity address frame that is itself part of the unmarked link.

8 Formal verification

The Lean 4/mathlib development at <https://github.com/sashakolpakov/StickNumbers> gives a conditional certificate of the principal knot-counting deductions. On the actual polygon configuration space it verifies the cubic/quartic wall construction, path and component logic, and, for $N \geq 5$, the explicit finite estimate behind the $N^{3N+o(N)}$ upper bound. It also verifies the permutation family, Garside-word bookkeeping, knot-transfer maps, the finite lower bound, the headline $\Theta(N \log N)$ law, and the eventual inequality $cN \log N \leq \log |\mathcal{K}_N|$ for every $c < 2/3$. Here the bounded-stick subtype belongs to a supplied knot-type interface, not to a formalized quotient by ambient isotopy. Generic support and isotopy invariance, Barone–Basu, Garside normal-form theory, Malyutin–Stupakov, Huh–Oh, and orientation reversal remain typed external theorems; the optional $-\frac{2}{3}N \log \log N + O(N)$ refinement has a separate arithmetic interface, and the auxiliary link construction is outside the formal target. The repository contains the declaration-level audit and build instructions.

9 Fewnomial and determinant directions

The crossing equations have two special features. First, each wall is a constant-support polynomial: the affine four-point determinant $\det(p_b - p_a, p_d - p_c, p_c - p_a)$ has at most 24 monomial terms. Second, the family of all such walls is highly correlated: these are determinant walls coming from one point configuration, not unrelated sparse cubics. This places the problem between fewnomial real algebraic geometry and the combinatorics of order types.

One possible route is determinant-combinatorial. The signs of all affine four-point determinants are the chirotope data of the labeled configuration in $\mathbb{R}^3$. Classical work of Goodman–Pollack gives $\exp(O(N \log N))$ -type bounds for fixed-dimensional point configurations [14]. Alon’s refinement gives the fixed-dimensional asymptotic order for configurations: for fixed d , the number of labeled order types of N points in $\mathbb{R}^d$ has the form

$$N^{d^2 N + o(N)}$$

[3]. The same work gives $N^{\Theta(N)}$ upper and lower bounds for fixed-rank real matroids. Thus affine order types in $\mathbb{R}^3$ live at scale $N^{9N + o(N)}$, while rank-four vector matroid data live at $N^{\Theta(N)}$. Our crossing discriminant uses only a structured subset of the four-point determinant data. This is why its count is much smaller than the count of all order types.

Gros–Ramirez Alfonsin’s strong geometry makes this comparison quantitative [15]. Homogenize the N vertices to vectors $v_i = (1, p_i) \in \mathbb{R}^4$. Their full wedge ground set consists of the normals to the hyperplanes spanned by ordered triples of the v_i , hence has at most N^3 elements. Its rank-four chirotope therefore has at most N^{12} sign entries. Replacing each unit normal by its unnormalized cofactor vector does not change these signs. One coordinate of such a normal has degree three in the vertex coordinates and the other three have degree at most two, so a wedge determinant has degree at most nine. A standard realizable-sign-pattern bound [23] in $3N$ variables consequently gives

$$\#\{\text{realizable full wedge tables}\} \leq N^{33N + o(N)}.$$

Their reconstruction theorem says that this is also an upper bound for generic polygonal-knot types, and a sufficiently small generic perturbation covers every type in $\mathcal{K}_N$.

One can do much better without counting their unused signs. In their proof, the crossing, over–under, and crossing-sign queries use $O(N^2)$ ordinary affine determinants. The ordering of crossings along a projected edge uses one witness sign for each triple of polygon edges, hence $O(N^3)$ signs of degree at most six. Applying the same sign-pattern estimate only to these queries gives the implicit bound

$$|\mathcal{K}_N| \leq N^{6N + o(N)}.$$

Theorem 2 improves this to $N^{3N + o(N)}$ by counting connected components of the complement of only $O(N^2)$ cubic and quartic self-intersection walls.

There is an important distinction between counting determinant sign data and counting chambers. Order-type and oriented-matroid bounds count possible sign patterns of minors. The chamber count asks for connected components of realization sets for such sign patterns, after intersecting with the stick-knot discriminant complement. Realization spaces of oriented matroids can be topologically complicated by universality phenomena [21, 24]; hence matroid enumeration alone does not automatically give the chamber count. Here it nevertheless bounds knot types because the strong-geometry theorem provides a direct map from the sign data to isotopy class. Our chamber method uses a smaller polynomial family and does not claim that its sign vector alone classifies the knot.

A second route is genuinely fewnomial. Existing fewnomial theorems control positive solutions of sparse systems and the topology of sparse hypersurfaces [17, 6, 5, 7]. The chamber problem here asks for connected components of sign conditions for $O(N^2)$ sparse determinant walls on the Stiefel manifold. The useful theorem would keep the dependence on the common sparse support and the determinant incidence structure, rather than replacing the equations by an arbitrary family of bounded-degree polynomials.

Question 1 (Sparse determinant arrangement problem). *Can one prove a sign-condition component bound for the sparse determinant-wall arrangement on $\text{St}_3(H)$ that improves the exponent or leading constant in the $N^{O(N)}$ bound of Theorem 2?*

There is also a lower-bound version of the same problem. Fixing a regular planar shadow, the over/under data are signs of height differences at its crossings, hence are constrained by a finite system of linear inequalities in the vertex heights. This gives a concrete feasibility problem for crossing assignments. If one can construct shadows for which many feasible assignments have distinct knot invariants, then the fewnomial/determinant chamber structure would also become a knot production mechanism, not only an upper-bound mechanism.

Acknowledgments

The authors used OpenAI’s GPT-5.6-sol as a research and coding assistant to help strengthen the results, improve the exposition, and generate and audit the Python and Lean 4 code accompanying this paper. The authors reviewed the resulting mathematics and code and take responsibility for the final content.

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